4. Long Wave

4.1 Long Wave Assumption

Let us consider "a long wave," that is, a wave in which the wavelength L is sufficiently long as compared with the depth D: $L >> D$. The non-vortex condition is then written as

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{1}$$

Definite integration with $y[\eta, -D]$ of (1) yields

$$u(\eta) - u(-D) = \frac{\partial v(y = \eta)}{\partial x} \times D \approx v(y = \eta) \times \frac{D}{L} \tag{2}$$

Hence, the relative difference in horizontal particle velocity is

$$\frac{u(\eta) - u(-D)}{u(\eta)} = \left[\frac{v}{u}\right]_{y=\eta} \times \frac{D}{L} \tag{3}$$

From the discussions in the previous chapter, we already know that $[v/u]_{y=\eta}$ has a maximum of unity; hence, the condition $L >> D$ implies that the difference between $u(\eta)$ and $u(-D)$ is relatively small, even if the absolute value of $u(\eta)$ is not always small.

Thus, we can consider that if the long wave condition is satisfied, the horizontal particle speed is approximately a constant.

For three-dimensional wave problems of long waves, we can also assume that the horizontal components of the water particle velocity are constant.

4.2 Long Wave Problems of Three-Dimensional Case

(a) Mass Conservation Equation

Next, let us consider three-dimensional long wave problems. We set the origin of a rectangular co-ordinate system as a point on the stable sea surface, and take the x-and y-axis horizontally (for example, the x-axis in the eastward direction, and the y-axis in the northward direction); the z-axis is taken perpendicular to the x- and y-axis. The mass conservation law is expressed as

$$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{4}$$

By performing a definite integration of (4) w.r.t. z within the region $z = [\eta, -D]$, we have

$$w(S) - w(B) = -(D + \eta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{5}$$

Here, $w(S)$ and $w(B)$ are the vertical components of the particle velocity at the sea surface and bed, respectively.

The kinematic sea surface and the sea bed conditions are expressed as

$$w(S) = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} \qquad \text{on} \quad z = \eta(x, y, t) \tag{6}$$

and

$$w(B) = -\frac{\partial D}{\partial t} - u\frac{\partial D}{\partial x} - v\frac{\partial D}{\partial y} \quad \text{on} \quad z = -D(x, y, t) \tag{7}$$

Let us assume that the depth D does not change with time; (7) then becomes

$$w(B) = -u\frac{\partial D}{\partial x} - v\frac{\partial D}{\partial y} \qquad \text{on} \quad z = -D(x, y) \tag{8}$$

By substituting (6) and (8) into (5), we have

$$\frac{\partial \eta}{\partial t} = -\left[\frac{\partial\{(D+\eta)u\}}{\partial x} + \frac{\partial\{(D+\eta)v\}}{\partial y}\right] \tag{9}$$

(Note: (5-6) are satisfied only at the sea surface ($z = \eta$), and (8) is satisfied only on the sea bed ($z = -D$). However, there are no z-dependent terms on the right-hand sides of both (5-6) and (8), and hence, we can consider that equations (6) and (8) are satisfied generally at any depth. Hence, no boundary condition is given in equation (9).

Equation (9) is the mass conservation condition for a long wave. Note that non-linearity is not assumed in its derivation.

(b) Equations of Motion

Since Bernoulli's formula is satisfied at any point in a liquid, at the position (x, y, z), it has the following form:

$$-\phi_t + \frac{1}{2}(u^2 + v^2) + gz + p(x, y, z) = F(t) \tag{10}$$

Further, the equation has the following form at a point on the sea surface:

$$-\phi_t + \frac{1}{2}(u^2 + v^2) + g\eta + P_0 = F(t) \tag{11}$$

Here, P_0 denotes the atmospheric pressure and is assumed to be a constant.

Subtracting (10) from (11), we have

$$p - P_0 = \rho g(\eta - z) \quad (12)$$

This shows that the pressure distribution takes the same form as that of the case of static pressure for no motion. On the other hand, the equation of motion in the x-direction is given by

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad (13)$$

Comparing (5-12) and (13), we have

$$\frac{Du}{Dt} = -g\frac{\partial \eta}{\partial x} \quad (14)$$

Similarly, we have the equation of motion in the y-direction as

$$\frac{Dv}{Dt} = -g\frac{\partial \eta}{\partial y} \quad (15)$$

Note that we have used only the long wave assumption until now and have not used the non-linear assumption.

(c) Linear approximation

Next, we consider dropping non-linear terms. We introduce flow (q_x, q_y) as the integrated value of the particle velocity (u, v), . Therefore,

$$q_x = \int_{-D}^{\eta} u dz = (\eta + D)u, \quad \text{and} \quad q_y = \int_{-D}^{\eta} v dz = (\eta + D)v \quad (16)$$

Equation (9) is then expressed simply as follows:

$$\frac{\partial \eta}{\partial t} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}\right) \quad (17)$$

It is possible to rewrite equations (5-14) and (5-15) in the following manner by using q_x, q_y with a non-linear approximation:

$$\frac{\partial q_x}{\partial t} = -gD\frac{\partial \eta}{\partial x} \quad (18)$$

and

$$\frac{\partial q_y}{\partial t} = -gD\frac{\partial \eta}{\partial t} \quad (19).$$

We differentiate (17) with respect to t, and substitute (18) and (19) in it; we then have

$$\frac{\partial^2 \eta}{\partial t^2} = g\left\{\frac{\partial}{\partial x}\left(D\frac{\partial \eta}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial \eta}{\partial y}\right)\right\} \tag{20}$$

Equation (20) is the long wave equation governing many problems in the studies on tsunami and storm surge. Note that we did not assume uniform depth in deriving (20), and hence, (20) is applicable for long waves propagating in seas with non-uniform depths.

In the case of a sea with a uniform depth, (20) can be re-written as follows:

$$\frac{\partial^2 \eta}{\partial t^2} = gD\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) \tag{21}$$

This is similar to the equation of wave motion, and the wave velocity c is given by

$$c = \sqrt{gD} \tag{22}$$

4.3 Long waves entering a V-shaped bay

We next consider the mechanism of a wave entering a V-shaped bay, as shown in the figure.

The innermost point is assumed to be the origin and the x-axis considered to be along the centerline of the bay extending toward the bay mouth.

We consider the problem to be a one dimensional in nature, and the width b is given by $b = ax$

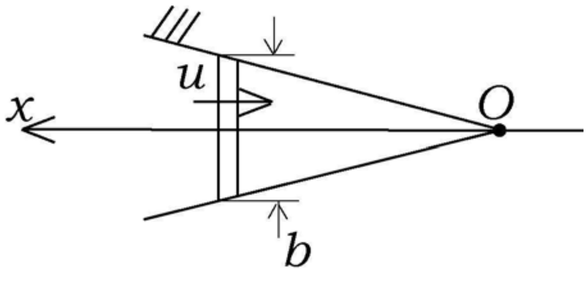

Fig 1. Tsunami wave going into a V-shaped bay

(Case A) A bay of constant depth

The equation of motion is given by

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{23}$$

The water pressure is the same as the static pressure as follows:

$$p = \rho g(\eta - z) \tag{24}$$

We substitute (24) into (23), and we have

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \tag{25}$$

The equation of mass conservation is ("flow" can be defined as $q(x) \equiv ubD$)

$$\frac{\partial}{\partial t}(b\eta dx) = -\frac{\partial}{\partial x}(ubD) \tag{26}$$

Since the width b of the bay is given by $b = ax$, we have

$$\frac{\partial \eta}{\partial t} = -D\left(\frac{\partial u}{\partial x} + \frac{u}{x}\right) \tag{27}$$

By eliminating u from (26) and (27), we have

$$\frac{\partial^2 \eta}{\partial x^2} + \frac{1}{x}\frac{\partial \eta}{\partial x} - \frac{1}{c^2}\frac{\partial^2 \eta}{\partial t^2} = 0 \tag{28}$$

Here $c^2 = gD$. Now, we substitute $\varsigma = Z(x)e^{-i\sigma t}$, and (28) becomes

$$Z'' + \frac{1}{x}Z' + \frac{\sigma^2}{c^2}Z = 0 \tag{29}$$

This equation (4-29) is the Bessel's function of the zero-th order.

[Mathematical notes]

I recommend that you regard the Bessel function as a cosine function with the amplitude attenuating proportional to $1/\sqrt{x}$, that is,

$$J_0(x) = \frac{1}{\sqrt{x}}\cos x \tag{30}$$

(this does not result in a significant difference from exactness). By doing this, you can easily understand the characteristics of a Bessel function.

Neumann function is a sin function with the amplitude attenuating proportional to $1/\sqrt{x}$, that is,

$$N_0(x) = \frac{1}{\sqrt{x}}\sin x \tag{31}$$

It is well known that e^{ix} and e^{-ix} is defined by the formula

$$e^{ix} = \cos x + i\sin x,\quad e^{-ix} = \cos x - i\sin x$$

Analogically, we introduce

$$J_0(x) + iN_0(x) = \frac{1}{\sqrt{x}}e^{ix} \tag{32}$$

This is called the Hankel function of the first kind, and is written as $H_0^{(1)}(x)$. Similarly, we define

$$J_0(x) - iN_0(x) = \frac{1}{\sqrt{x}} e^{-ix} \tag{33}$$

as the Hankel's function of the second kind, and we write it as $H_0^{(2)}(x)$.

[Conclusion] Thus, Bessel and Hankel functions $J_0, N_0, H_0^{(1)}, H_0^{(2)}$ are $\cos x, \sin x, e^{ix}, e^{-ix}$ with their amplitude attenuating proportionally with x^2.

Hopefully, your "Bessel function allergy" must be cured by now. In exact terms, Bessel's functions should be written as

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$$J_0(x) = \sqrt{\frac{2}{\pi x}} \cos(x + \frac{\pi}{4}), \quad N_0(x) = \sqrt{\frac{2}{\pi x}} \sin(x + \frac{\pi}{4})$$

However, both the constant coefficient $\sqrt{2/\pi}$ and the phase shift $\pi/4$ are not essential.

The solution of (30) can be rewritten as

$$Z(x) = \{J_0, N_0, H_0^{(1)}, H_0^{(2)}\}(kx) \tag{30'}$$

We can freely select any linear combination of two functions out of four in {}, similar to the way in which the solution of $y'' + 9y = 0$ is written as $y = \{\cos 3x, \sin 3x, e^{3ix}, e^{-3ix}\}$.

Since the incoming wave moves in the negative x-axis direction, the solution must contain the factor $e^{-i(kx+\sigma t)}$. Hence, we select a function that contains the factor $\frac{1}{\sqrt{x}} e^{-ikx}$ Further, this is a Hankel function of the second kind. Hence, we finally decide the (exact) solution of the present problem as

$$\eta = AH_0^{(2)}(kx)e^{-i\sigma t} \tag{31-a}$$

Question: What type of waves are indicated by the form of the following solutions:

$$\eta = AH_0^{(1)}(kx)e^{-i\sigma t} \tag{31-b}$$

$$\eta = AJ_0(kx)e^{-i\sigma t} \tag{31-c}$$

and

$$\eta = AN_0(kx)e^{-i\sigma t} \tag{31-d}$$

Equation (31-a) shows that the amplitude of the long wave (a tsunami) is inversely proportional to the square root of the distance from the innermost point. Further, the energy of the waves changes proportionally with the square of the amplitude. It is natural the amplitude is inversely proportional to the distance from the origin because the change in energy density is inversely proportional to the distance from the original point in the present case.

(Case B) A long wave entering a V-shaped bay of a sloping sea bed

Next, we consider the case of a long wave entering a V-shaped bay, where the depth changes linearly from the innermost point.

[Solution] If the width is given by $b = ax$, and the depth is given by $D = mx$, then we finally obtain

$$\frac{\partial^2 \eta}{\partial x^2} + \frac{2}{x}\frac{\partial \eta}{\partial x} - \frac{1}{gmx}\frac{\partial^2 \eta}{\partial t^2} = 0 \qquad (32)$$

[Students' Homework] Derive (32) and solve it.

[Hint] The solution of

$$Z'' + \frac{1-2\alpha}{x}Z' + \left[\left(\beta\gamma x^{\gamma-1}\right)^2 + \frac{\alpha^2 - \nu^2\gamma^2}{x^2}\right]Z = 0 \qquad (33)$$

is given by the linear combination of the following Bessel's functions

$$Z = x^{\alpha} \times \left\{J_\nu, N_\nu, H_\nu^{(1)}, H_\nu^{(2)}\right\}(\beta x^{\gamma}) \qquad (34)$$

[Suugaku-Koshiki, Tables of Mathematical Formulae, Iwanami, vol. 3, p161]

[Case C]

Problem: Constant width (a rectangular bay) and the depth changing linearly as $D = mx$. Show the following result for this case.

$$\eta = H_0^{(2)}(2\sqrt{k}x)e^{-i\sigma t} \qquad (35)$$

Why the function should be selected Hankel's function of the second kind, and we should neither select Bessel's function $J_0(2\sqrt{k}x)$ or Hankel's function of the first kind $H_0^{(1)}(2\sqrt{k}x)$.

[Case D] Exiting **wave**

This is the case that the wave source is in a small bay on a straight line coast, and the waves leave the shore from the bay mouth. If the depth of the open sea is a constant, the radiated wave is expressed as

$$\eta = AH_0^{(1)}(kr)e^{-i\sigma t} \tag{36}$$

Prove that (36) is a solution of an exiting wave of this case

4.4 Numerical Calculation of the Propagation of a Long Wave (a Tsunami)

Tsunami propagation (simulation) is studied numerically by integrating three equations, (17)-(19), with respect to time considering sea bottom friction.

$$\frac{\partial q_x}{\partial t} = -gD\frac{\partial \eta}{\partial t} - f_c\frac{q_x\sqrt{q_x^2+q_y^2}}{D^2} - \left\{\frac{q_x}{D}\frac{\partial}{\partial x} + \frac{q_y}{D}\frac{\partial}{\partial y}\right\}q_x \tag{38}$$

$$\frac{\partial q_y}{\partial t} = -gD\frac{\partial \eta}{\partial y} - f_c\frac{q_y\sqrt{q_x^2+q_y^2}}{D^2} - \left\{\frac{q_x}{D}\frac{\partial}{\partial x} + \frac{q_y}{D}\frac{\partial}{\partial y}\right\}q_y \tag{39}$$

$$\frac{\partial \eta}{\partial t} = -\left\{\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}\right\} \tag{40}$$

The second terms of (38) and (39) express the influences of bottom friction, and f_c is a dimensionless value of the friction coefficient on the sea bed, which generally has a value ranging from 0.005 to 0.03. This term can be neglected at the sea area where the sea depth D is larger than 100 m.

The third terms in (38) and (39) are the terms for the non-linear effects, which can be also neglected in a deeper sea area where the ratio of the amplitude to the depth $|\eta|/D$ does not exceed a few percents.

(a) Initial conditions given for a numerical calculation of a tsunami propagation model

If we obtain a set of fault parameters of a submarine earthquake, we can calculate the distribution of the vertical component of the sea bottom deformation, or the vertical crustal motion At an initial time $t = 0$, we give the distribution of the sea surface displacement η the same pattern as the sea bottom deformation.

(b) $q_x, q_y = 0$ at the initial time.

(c) We select a time interval τ and rewrite (3-58) through (3-40) to a set of difference equation , and by applying the "leap frog method ", we obtain

$$\begin{aligned}\Delta q_{x,i,j} &= -g(D_{i+1,j}+D_{i,j})/2\times(\eta_{i+1,j}-\eta_{i,j})\tau/\Delta l\\ \Delta q_{y,i,j} &= -g(D_{i,j+1}+D_{i,j})/2\times(\eta_{i,j+1}-\eta_{i,j})\tau/\Delta l\\ \Delta\eta_{i,j} &= -\{(q_{x,i,j}-q_{x,i-1,j})+(q_{y,i,j}-q_{y,i,j-1})\}\tau/\Delta l\end{aligned} \qquad (41\text{-a,b,c})$$

where Δl is the size of the grid interval.

By applying this method, we can numerically calculate the propagation of tsunami waves.

In real situations, we must consider two additional factors:

(1) Selection of the time step τ, and

(2) Two types of area boundaries:

2-1. Coast line boundary

2-2. Open boundary

(1) Selection of the time interval τ

For performing numerical calculations, we must consider the stability condition of the process of numerical calculation of a "leap frog type" method,which is called CFL condition: the time interval τ should not exceeded the following limit:

$$\tau \le \Delta l/\sqrt{2gD_{\max}} \qquad (42)$$

Here, $D_{\max}$ is the maximum depth in the considered sea area.

(2) Area boundaries

2-1. Coast line boundary

This condition can be realized by using $q_x = 0$ or $q_y = 0$ at the coast line boundary, or by substituting

$$\eta_{i+1,j} \equiv \eta_{i,j} \qquad (43)$$

where $(i+1, j)$ is the grid beyond the coast line. If we do this, the boundary plays the role of a mirror surface, and the perfect reflection condition will be satisfied.

2.2. Open boundary

We should set a free exiting wave condition artificially at the free boundary, where we should make the wave leave the area naturally. In order to satisfy this, we give the equation of the flow as follows:

$$q_x = \pm\sqrt{gD\eta^2 - q_y^2} \tag{44}$$

for east (+) or west (–) boundaries, and

$$q_y = \pm\sqrt{gD\eta^2 - q_x^2} \tag{45}$$

for the north (+) or south (-) boundaries.

[Note] We first consider the one-dimensional case: waves move eastward in a uniform depth channel

The equation of motion is given as follows:

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \tag{46}$$

Further, the equation of mass conservation is given as follows:

$$\frac{\partial \eta}{\partial t} = -D\frac{\partial u}{\partial x} \tag{47}$$

Eliminating u, we have

$$\frac{\partial^2 \eta}{\partial t^2} = gD\frac{\partial^2 \eta}{\partial x^2} \tag{48}$$

This is an equation of wave motion of the one-dimensional case and has the following solution:

$$\eta = f_1(x-ct) + f_2(x+ct) \tag{49}$$

Here $c^2 = gD$.

We already had studied that $f_1(x-ct)$ is the right (or eastward) wave component and $f_2(x+ct)$ is the leftward wave component. If we consider only the component of an eastward wave, it satisfies

$$\eta = f_1(x-ct)\,. \tag{50}$$

We note that (5-48) is the solution of the following partial differential equation.

$$\frac{\partial \eta}{\partial t} + c\frac{\partial \eta}{\partial x} = 0 \quad \text{or} \quad \frac{\partial \eta}{\partial t} = -c\frac{\partial \eta}{\partial x} \tag{51}$$

Along with (51) and (47), we have

$$-D\frac{\partial u}{\partial x} = -c\frac{\partial \eta}{\partial x} \tag{52}$$

This is integrated with respect to x and becomes simply (note that $Du = q_x$)

$$q_x = c\eta \tag{53}$$

This is the condition of a freely exiting wave toward the eastern boundary.

If the direction of wave propagation is oblique to the boundary, we should re-write (5-51) as follows:

$$q=\sqrt{q_x^2+q_y^2}=c\eta \tag{54}$$

It is possible to derive (44) and (45) from equation (54).

(3) Influence of the curvature of the Earth's surface

If the size of the calculation area does not exceed 1000 km × 1000 km, then we can neglect the influence of the curvature of the Earth's surface. However, if the size of the calculation mesh grid exceeds 1000 km, the influence of the Earth's surface should be considered. For such a case, instead of the set of equations (38) through (40), we can use the following set of equations:

$$\begin{aligned}\frac{\partial q_\lambda}{\partial t}&=-\frac{gD}{R\cos\phi}\frac{\partial\eta}{\partial\lambda}+Fq_\phi\\ \frac{\partial q_\phi}{\partial t}&=-\frac{qD}{R}\frac{\partial\eta}{\partial\phi}-Fq_\lambda\end{aligned} \tag{55a,b}$$

where F is Coriolis' coefficient having a value of $4\pi/(60\times 60\times 24)\times\sin\phi$ and λ and ϕ are the longitude and latitude of each grid, respectively; R is the radius of Earth. For such a large-scale grid, we cannot neglect the influence of the rotation of the earth. The mass conservation equation is given by

$$\frac{\partial\eta}{\partial t}=-\frac{1}{R\cos\phi}\left(\frac{\partial q_\lambda}{\partial\lambda}+\frac{\partial\left(q_\phi\cos\phi\right)}{\partial\phi}\right) \tag{56}$$

(4) Calculation of grid nesting

What should the size of the grid mesh be for performing numerical calculations of the propagation of a tsunami wave? Shuto (1984) conducted several numerical experiments and concluded that we should prepare more than 20 grid points for one period of a wave.

In such a case, we should prepare finer meshes for shallower sea regions than for deeper ones because, the wavelength becomes shorter in inversely proportion to , the square root of the depth (Why?).

Thus, we prepare several sets of nesting meshes, with the finer ones being for shallower regions.

In general, a grid interval of 1 km mesh for a deeper sea, 0.5 km for the shelf area, 0.25 km, 0.125km for the sea area approaching a tide gauge port, 62.5m for the coastal area are used.